MULTIPLICATIVE INVARIANTS AND SEMIGROUP ALGEBRAS

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ABSTRACT. Let G be a finite group acting by automorphism on a lattice A, and hence on the group algebra $S = k[A]$. The algebra of G-invariants in S is called an algebra of multiplicative invariants.

We investigate when algebras of multiplicative invariants are semigroup algebras. In particular, we present an explicit version of a result of Farkas stating that multiplicative invariants of finite reflection groups are indeed semigroup algebras. On the other hand, multiplicative invariants arising from fixed point free actions are shown to never be semigroup algebras. In particular, this holds whenever G has odd prime order.

INTRODUCTION

This article continues our investigaton of multiplicative invariants in [16, 17, 18] and is motivated by Farkas' work in [4, 5, 6].

Our specific focus here is a suitable permanence theorem for multiplicative actions of finite groups analogous to the classical Shephard-Todd-Chevalley Theorem for "linear" actions of finite groups (of good order) on polynomial algebras; this theorem states precisely when the corresponding algebra of invariants is again a polynomial algebra (e.g., [1, p. 115]).

Multiplicative actions, also called exponential actions [1], are certain group actions on Laurent polynomial rings or, equivalently, group algebras of lattices. Specifically, let A denote a lattice, i.e., a free abelian group of finite rank, and let G be a group acting by automorphisms on A. This action extends uniquely to an action of G on the group algebra $S = k[A]$ of A. Actions of this type are referred to as *multiplicative actions*, and the resulting algebra of invariants $R = S^G$ is called an algebra of *multiplicative invariants*. It is easy to see that R is again a group algebra only if G acts trivially on A; see (2.3). Thus the permanence theorem we have in mind is a characterization of all multiplicative actions yielding invariants that are semigroup algebras. Unfortunately, this article falls short of reaching this goal.

Here is the state of affairs and our contribution. It is implicit in [5, proof of Theorem 10] that multiplicative invariants of finite reflection groups are indeed semigroup algebras; this has been pointed out by Farkas himself in [6, p. 72]. After deploying

the requisite background material and some technicalities in sections 1 and 2, we present in section 3 an explicit proof of Farkas' theorem, along with an analysis of the structure of the corresponding semigroup and of the class group of the invariant algebra. The result, Theorem (3.3), is derived from a classical fact [1, Théorème 1 on p. 188] concerning multiplicative invariants of Weyl group actions on weight lattices of root systems. The method employed leads directly to an explicit fundamental system of invariants. I don't know if the converse of Farkas' theorem holds: Do all multiplicative invariants that are semigroup algebras come from reflection groups? Section 4 offers a first step in this direction. We show in Theorem (4.3) that multiplicative invariants of fixed point free actions (in rank at least 2) are never semigroup algebras. In particular, this holds for all multiplicative actions of finite groups of odd prime order. Our main tool in this section is an investigation of the singularities of multiplicative invariants. Doubtless, a good deal more can be said on this topic. Ultimately, the complete proof of the envisioned converse to Farkas' theorem will likely involve an investigation of certain group actions on semigroup algebras rather than just group algebras, and this will actually presumably be the proper setting for the theorem.

We hope to return to these questions in a future publication.

Notations and Conventions. Throughout this note, k will denote a commutative field. All monoids considered in this article are understood to be commutative. We use $\mathbb{Z}_+$ to denote the set of nonnegative integers and similarly for $\mathbb{R}_+$. Further notation will be introduced below, in particular in (2.1).

1. SEMIGROUP ALGEBRAS

1.1. **Commutative semigroup algebras.** Let M denote a monoid, with operation written as multiplication and identity element 1, and let $k[M]$ denote the semigroup (or monoid) algebra of M over k. Thus every element $\alpha \in k[M]$ can be uniquely written in the form

$$\alpha = \sum_{m \in M} k_m m \quad \text{with } k_m \in k \text{ almost all zero.}$$

The set $\{m \in M \mid k_m \neq 0\}$ is called the *support* of α, denoted $\text{Supp}(\alpha)$. Multiplication in $k[M]$ is defined by k-linear extension of the multiplication of M.

A good reference for general ring theoretic properties of commutative semigroup algebras is [9]. We note in particular the following facts:

- The k-algebra $k[M]$ is finitely generated (*affine*) if and only if M is a finitely generated monoid. This is trivial.
- $k[M]$ is a domain iff M is *cancellative* ($ax = ay \Rightarrow x = y$ for $a, x, y \in M$) and *torsion-free* ($x^n = y^n$ ($n > 0$) $\Rightarrow x = y$ for $x, y \in M$); see [9, Theorem 8.1].

- Assume $k[M]$ is a domain. Then $k[M]$ is integrally closed iff M is *normal*: $x^n = y^n z$ for $x, y, z \in M$ implies $z = z_1^n$ for some $z_1 \in M$; see [9, Corollary 12.11].

1.2. **Affine normal semigroups.** Finitely generated cancellative torsion-free normal monoids are often simply referred to as *affine normal semigroups*. By (1.1), we have for any monoid M:

> *The k-algebra $k[M]$ is an affine integrally closed domain iff M is an affine normal semigroup.*

As a reference for affine semigroup algebras in particular, I recommend [3]. By [3, Proposition 6.1.3], affine normal semigroups M have the following structure:

> $M = \mathrm{U}(M) \times M_+$, *where* $\mathrm{U}(M)$, *the group of units of M, is a free abelian group of finite rank and M_+ is an affine normal semigroup that is* positive, *that is,* $\mathrm{U}(M_+) = \{1\}$.

Further, by [9, Theorem 11.1], the group of units of $k[M]$ is given by:

$$\mathrm{U}(k[M]) = k^* \times \mathrm{U}(M) \ .$$

The k-algebra map $\mu : k[M] \to k$ that is given by $\mu(m) = 1$ for $m \in \mathrm{U}(M)$ and $\mu(m) = 0$ if $1 \neq m \in M_+$ is called the *distinguished augmentation* of $k[M]$.

1.3. **Gubeladze's polytope.** Let M be a finitely generated monoid that is cancellative and torsion-free, and let $\mathrm{gp}(M)$ denote the group of fractions of M. Our hypotheses on M imply that $\mathrm{gp}(M)$ is free abelian of finite rank and the natural map $M \to \mathrm{gp}(M)$ is an embedding. Thus M is embedded in the real vector space $V = \mathbb{R} \otimes \mathrm{gp}(M)$. Denote by $C(M)$ the convex cone in V that is spanned by M; so (using additive notation in V),

$$C(M) = \mathbb{R}_+ M = \{r_1 m_1 + \cdots + r_k m_k \mid r_i \in \mathbb{R}_+, m_i \in M\} \ .$$

Assume now that M is positive, that is, $\mathrm{U}(M)$ is trivial. Then there exists an affine hyperplane H in V with $0 \notin H$ such that $C(M)$ is the convex cone in V that is spanned by

$$\Phi(M) = H \cap C(M) \ .$$

The set $\Phi(M)$ is a polytope (the convex hull of finitely many points) in H. Different choices of H lead to projectively equivalent polytopes; see [10], [11].

Following Gubeladze [11], M is called Φ-*simplicial* iff $\Phi(M)$ is a simplex (the convex hull of finitely many affinely independent points). For several equivalent characterizations of this notion, see [11, Proposition 1.1].

1.4. **Torus action.** Let M be an affine normal semigroup. Then, as we observed above, M embeds into the lattice $L = \text{gp}(M) \cong \mathbb{Z}^n$ for some n. Assuming k algebraically closed for now, the algebraic torus $T = \text{Hom}(L, k^*) \cong (k^*)^n$ operates on $k[M]$ via

$$m^\tau = \tau(m)^{-1}m \qquad (\tau \in T, m \in M)$$

and k-linear extension to all of $k[M]$. This operation is rational, and the corresponding operation on $\text{Max}\, k[M] = \text{Hom}(k[M], k)$ has the distinguished augmentation as its only fixed point if $\text{U}(M)$ is trivial, and no fixed points at all otherwise. For further background on the geometric aspect of affine normal semigroup algebras, see [8]; the above assertion about fixed points is an (easy) exercise in [8, p. 28].

2. Multiplicative Invariants

2.1. **Basics.** The following notation will be kept throughout this article:

A	will be a free abelian group of finite rank;
$S = k[A]$	will denote the group algebra of A over k;
G	will be a finite group acting be automorphisms on A, and hence on S as well; the action will be written exponentially, $a \mapsto a^g$;
$R = S^G$	is the subalgebra of G-invariants in S.

In this situation, A is often called a G-*lattice.* As our main concern is R, the algebra of multiplicative G-invariants, we may assume that the G-lattice A is *faithful*, that is, the map $G \to \text{GL}(A)$ that defines the G-action is injective. Finally, A will be called *effective* if the subgroup A^G of G-invariant elements of A is trivial.

The *orbit sum* of an element $a \in A$ is the element of S that is defined by

$$\sigma(a) = \sum_{x \in a^G} x \ \in S \ .$$

where $a^G = \{a^g \mid g \in G\} \subseteq A$ denotes the G-orbit of a. Orbit sums are clearly G-invariant, and hence they actually belong to R. In fact, they provide a k-basis for R :

$$R = \bigoplus_{a \in A/G} k\sigma(a) \ ,$$

where A/G denotes a transversal for the G-orbits in A. As k-algebra, R is an affine integrally closed domain; all these properties are inherited from S. (Note that A is an affine normal semigroup.)

2.2. **Passage to an effective lattice.** Let $^{-}$ denote the canonical map $A \twoheadrightarrow A/A^G$ and its extension to S; so

$$^{-}: S = k[A] \twoheadrightarrow \overline{S} = k[A/A^G] , \quad a \mapsto aA^G \ (a \in A) .$$

Note that $\overline{A} = A/A^G$ is a G-lattice and the map $^{-}$ is G-equivariant. Moreover, letting G_x denote the isotropy (stabilizer) subgroup of G of an element x in A or in $\overline{A}$, we have

(1) $$G_a = G_{\overline{a}} \quad \text{for all } a \in A.$$

Here, the inclusion $G_a \subseteq G_{\overline{a}}$ is clear. The reverse inclusion follows from the fact that the map $G_{\overline{a}} \to A^G$, $g \mapsto a^g a^{-1}$, is a group homomorphism, and hence it must be trivial, as $G_{\overline{a}}$ is finite while A^G is torsion free. We deduce from the above equality of isotropy groups that

$\overline{A}$ *is an effective G-lattice.*

Further, $^{-}: S \to \overline{S}$ sends the orbit sum $\sigma(a)$ to the orbit sum $\sigma(\overline{a})$, and $\sigma(\overline{a}) = \sigma(\overline{b})$ is equivalent with $\sigma(a) = \sigma(b)c$ for some $c \in A^G$. Consequently,

The map $^{-}$ maps R onto the G-invariants in $\overline{S}$, that is, $\overline{R} = \overline{S}^G$. The kernel of this epimorphism is the ideal $(a - 1 \mid a \in A^G)$ of R.

Finally, every G-equivariant homomorphism from A to some effective G-lattice clearly factors through $^{-}: A \to \overline{A}$.

2.3. **Multiplicative invariants that are semigroup algebras.** In this section, we note some consequences of the assumption that R is a semigroup algebra. In particular, it will turn out that $\overline{R}$ is a semigroup algebra as well in this case.

Proposition. *Assume that $\varphi : k[M] \xrightarrow{\cong} R$ for some semigroup algebra $k[M]$. Then M is an affine normal semigroup, so $M = \mathrm{U}(M) \times M_+$ as in (1.2), and M_+ is Φ-simplicial. Moreover, the isomorphism φ can be chosen so that $\varphi(\mathrm{U}(M)) = A^G$. Finally, φ restricts to an isomorphism $k[M_+] \xrightarrow{\cong} \overline{R}$, in the notation of (2.2).*

Proof. First, M must be an affine normal semigroup, since R is an affine integrally closed domain; see (1.1) and (2.1). Hence, $M = \mathrm{U}(M) \times M_+$ and

$$k^* \times \mathrm{U}(M) = \mathrm{U}(k[M]) \cong \mathrm{U}(R) = \mathrm{U}(S)^G = k^* \times A^G .$$

Letting $\alpha : S \to k$ denote the distinguished augmentation of $S = k[A]$, sending all $a \in A$ to 1, the given isomorphism φ can be modified by defining $\psi(m) = \alpha\varphi(m)^{-1}\varphi(m)$ for $m \in \mathrm{U}(M)$ and $\psi(m) = \varphi(m)$ for $m \in M^+$ to obtain a new isomorphism $\psi : k[M] \xrightarrow{\cong} R$ which maps $\mathrm{U}(M)$ onto A^G. The composite $^{-} \circ \psi : k[M] \xrightarrow{\cong} R \twoheadrightarrow \overline{R}$ has kernel $\psi^{-1}\left((a - 1 \mid a \in A^G)\right) = (m - 1 \mid m \in \mathrm{U}(M))$. Hence, this map restricts to an isomorphism $k[M_+] \xrightarrow{\cong} \overline{R}$.

It remains to show that M_+ is Φ-simplicial. Now, M_+ is Φ-simplicial if and only if $k[M_+]$ is almost factorial, that is, the class group $\mathrm{Cl}(k[M_+])$ is torsion; see [11,

Proposition 1.6]. However, $\mathrm{Cl}(k[M_+]) \cong \mathrm{Cl}(k[M])$, by [7, Corollary 7.3 and Theorem 8.1]. Therefore, $\mathrm{Cl}(k[M]) \cong \mathrm{Cl}(R)$. Since $\mathrm{Cl}(S)$ is trivial, it follows form Samuel's theory of Galois descent (cf. [7, Theorem 16.1]) that $\mathrm{Cl}(R)$ embeds into $\mathrm{H}^1(G, \mathrm{U}(S)) = \mathrm{Hom}(G, k^*) \oplus \mathrm{H}^1(G, A)$, a finite $|G|$-torsion group. (The precise form of class groups of multiplicative invariants is known [16], but this information is not needed here.) We deduce that $\mathrm{Cl}(k[M_+])$ is finite $|G|$-torsion, thereby completing the proof.   □

As a very simple special case, assume that R is actually a group algebra; so $M = \mathrm{U}(M)$. Then the above proposition yields that $R = k[A^G]$. Now S is integral over $R = S^G$ and, on the other hand, A/A^G is torsion-free. Thus we must have $A = A^G$, whence G acts trivially on A. This substantiates a remark made in the introduction.

2.4. **A reduction lemma.** In this section, we will prove a technical lemma stating that an algebra of multiplicative invariants is a semigroup algebra provided a closely related one is. Let

$$\mathcal{M}(A)$$

denote the submonoid of $(R, \cdot)$ that is generated by the orbit sums $\sigma(a)$ for $a \in A$, and similarly for other G-lattices.

Lemma. *Let $A \subseteq B$ be G-lattices such that B/A is G-trivial. Suppose that $k[B]^G = kC$, the k-linear span of some subset $C \subseteq \mathcal{M}(B)$. Then $k[A]^G = kD$ with $D = C \cap k[A]$.*

Proof. Note that D is a subset of $k[A]^G$; so clearly $kD \subseteq k[A]^G$. For the other inclusion, let $\alpha \in k[A]^G$ be given. Then $\alpha = \sum_{c \in C} k_c c$ with $k_c \in k$ almost all zero. We show by induction on the minimum number, $n(\alpha)$, of nonzero terms in such an expression that $\alpha \in kD$. The case $n(\alpha) = 0$ (i.e., $\alpha = 0$) being obvious, assume $\alpha \neq 0$. Then some $d \in C$ with $k_d \neq 0$ must satisfy $\mathrm{Supp}(d) \cap A \neq \emptyset$. Say $d = \sigma(b_1) \cdot \ldots \cdot \sigma(b_l)$ with $b_j \in B$. Then

$$\mathrm{Supp}(d) \subseteq \{b_1^{g_1} \cdot \ldots \cdot b_l^{g_l} \mid g_j \in G\} \ .$$

So some product $b_1^{g_1} \cdot \ldots \cdot b_l^{g_l}$ belongs to A. Inasmuch as B/A is G-trivial, all these products are congruent to each other modulo A, and hence they all belong to A. Thus, $\mathrm{Supp}(d) \subseteq A$ and so $d \in D$. Since $\alpha - k_d d$ belongs to kD, by induction, we conclude that $\alpha \in kD$ as well. This proves the lemma.   □

Note that if the subset C in the Lemma is k-independent or multiplicatively closed then so is $D = C \cap k[A]$. Hence, if $k[B]^G = kC$ is a semigroup algebra, with semigroup basis C, then $k[A]^G = kD$ is a semigroup algebra with semigroup basis D.

We also remark, for future use, that the argument in the proof of the Lemma shows that, for $d = \prod_{j=1}^l \sigma(b_j) \in \mathfrak{M}(B)$,

$$(2) \qquad \prod_{j=1}^l \sigma(b_j) \in k[A] \iff \mathrm{Supp}(d) \cap a \neq \emptyset \iff \prod_{j=1}^l b_j \in A \ .$$

3. Reflection Groups

3.1. **Reflections.** An endomorphism ϕ of a vector space is called a *pseudoreflection* if $\mathrm{Id} - \phi$ has rank 1; ϕ is a *reflection* if, in addition, $\phi^2 = \mathrm{Id}$.

Keeping the notation of (2.1), we will assume in this section that A is a G-lattice which, without essential loss, will be assumed faithful. We will further assume that G is a reflection group on A; so:

G *is a finite subgroup of* $\mathrm{GL}(A)$ *that is generated by reflections.*

Here, an element $g \in G$ is called a *reflection* if g is a reflection on $A \otimes_\mathbb{Z} \mathbb{Q}$. We remark that, since $\det g = \pm 1$ holds for all $g \in G$, pseudoreflections in G are automatically reflections. They can also be characterized by the condition that the subgroup $A^{\langle g \rangle} = \mathrm{Ker}_A(g - \mathrm{Id})$ of g-fixed points in A have rank equal to $\mathrm{rank}(A) - 1$ or, alternatively,

$\mathrm{Ker}_A(g + \mathrm{Id}) = \{a \in A \mid a^g = a^{-1}\}$ *is infinite cyclic.*

As in (2.2), we let $^-$ denote the canonical map $A \twoheadrightarrow \overline{A} = A/A^G$. Note that (2.2)(1) implies that $\overline{A^{\langle g \rangle}} = \overline{A}^{\langle g \rangle}$ holds for all $g \in G$. Therefore, if g acts as a reflection on A then it does so on $\overline{A}$ as well, and conversely.

3.2. **Root systems.** Embed A into the $\mathbb{R}$-vector space $V = A \otimes_\mathbb{Z} \mathbb{R}$ and view G as a subgroup of $\mathrm{GL}(V)$. As is customary, we will use additive notation in A and V. Define

$$\rho(v) = |G|^{-1} \sum_{g \in G} v^g \qquad (v \in V) \ .$$

Thus, ρ is an idempotent $\mathbb{R}[G]$-endomorphism of V with $\rho(V) = V^G$, the subspace of G-fixed points in V. Putting $\pi = 1 - \rho \in \mathrm{End}_{\mathbb{R}[G]}(V)$, we obtain

$$A \subseteq \rho(A) \oplus \pi(A) \subseteq \rho(V) \oplus \pi(V) = V \ .$$

For each reflection $g \in G$, let the two possible generators of $\mathrm{Ker}_A(g + \mathrm{Id})$ be denoted $\pm a_g$. Define

$$\Phi = \Phi_{A,G} = \{\pm a_g \mid g \text{ a reflection in } G\} \ .$$

The crucial properties of Φ are listed in the following lemma due to Farkas [5, Lemmas 1–3].

Lemma. $\Phi = \Phi_{A,G}$ *is a reduced crystallographic root system in* $\pi(V)$, *and the restriction of* G *to* $\pi(V)$ *is the Weyl group* $\mathcal{W}(\Phi)$ *of* Φ. *Furthermore,*

$$\mathbb{Z}\Phi \subseteq A \subseteq \pi^{-1}(\Lambda) \ ,$$

where $\mathbb{Z}\Phi$, *the* $\mathbb{Z}$-*span of* Φ *in* V, *is the* root lattice *and* $\Lambda = \Lambda_{A,G} = \{v \in \pi(V) \mid v - v^g \in \mathbb{Z}a_g$ *for all reflections* $g \in G\}$ *is the* weight lattice *of* Φ.

For background on root systems, we refer to [1] or [12].

3.3. **Multiplicative invariants of reflection groups.** Our goal here is to prove the following result implicit in the work of Farkas [5, 6]. We will use the notation of (3.2).

Theorem. *Let A be a free abelian group of finite rank, and let G be a finite subgroup of $\mathrm{GL}(A)$ that is generated by reflections. Then the invariant algebra $R = k[A]^G$ is a semigroup algebra; in fact, $R \cong k[M]$ with $M = A^G \times (\pi(A) \cap \Lambda_+)$, where Λ_+ is the semigroup of dominant weights for some base of the root system $\Phi_{A,G}$.*

Proof. Fix a base $\Delta = \{\alpha_1, \dots, \alpha_r\}$ for $\Phi = \Phi_{A,G}$, i.e., Δ is a subset of Φ that is an $\mathbb{R}$-basis of $\pi(V)$ and such that $\Phi \subseteq \mathbb{Z}_+\Delta \cap -\mathbb{Z}_+\Delta$. So $\alpha_i = \pm a_{g_i}$ for certain reflections $g_i \in G$. The *fundamental dominant weights* $\lambda_1, \dots, \lambda_r$ are determined by $\lambda_i - \lambda_i^{g_j} = \delta_{i,j}\alpha_j$ (Kronecker delta); they form a $\mathbb{Z}$-basis of the weight lattice Λ. The semigroup Λ_+ of dominant weights for Δ is

$$\Lambda_+ = \oplus_{i=1}^r \mathbb{Z}_+\lambda_i .$$

It is a classical result [1, Théorème 1 on p. 188] that $k[\Lambda]^G$ is a polynomial algebra, with the orbit sums of the fundamental dominant weights as independent generators. In other words,

$k[\Lambda]^G = kE$, *with* $E = \langle \sigma(\lambda_1), \dots, \sigma(\lambda_r) \rangle \cong \Lambda_+$ *a k-independent submonoid of* $\mathcal{M}(\Lambda)$.

Now put $B = \rho(A) \oplus \Lambda$, a G-lattice in V with $A \subseteq B$ and B/A G-trivial. To see the latter, note that A contains $A^G \oplus \mathbb{Z}\Phi$, and $B/(A^G \oplus \mathbb{Z}\Phi) \cong \big(\rho(A)/A^G\big) \oplus (\Lambda/\mathbb{Z}\Phi)$ is G-trivial, since $\rho(A) \subseteq V^G$ and the Weyl group G of Φ acts trivially on the fundamental group $\Lambda/\mathbb{Z}\Phi$ of Φ; cf. [1, p. 167]. Inasmuch as $k[B] = k[\rho(A)] \otimes_k k[\Lambda]$, with $\rho(A) = B^G$, the G-invariants in $k[B]$ are given by $k[B]^G = k[B^G] \otimes_k k[\Lambda]^G$. Thus, using the above description of $k[\Lambda]^G$,

$$k[B]^G = k[B^G] \otimes_k kE = kC \quad \text{with} \quad C = B^G \times E .$$

Note that C is a k-independent submonoid of $\mathcal{M}(B)$. Lemma (2.4) therefore implies that $k[A]^G = kD$ is a semigroup algebra, with semigroup basis $D = C \cap k[A]$. It remains to verify the description of the monoid given in the theorem. To this end, note that, by (2.4)(2), the isomorphism $B^G \oplus \Lambda_+ \overset{\cong}{\longrightarrow} B^G \times E = C$ restricts to an isomorphism $M := (B^G \oplus \Lambda_+) \cap A \overset{\cong}{\longrightarrow} D$. Furthermore, writing $a \in A$ as $a = \rho(a) + \pi(a)$, we see that $a \in M$ if and only if $\pi(a) \in \Lambda_+$. Since $\mathrm{Ker}_A(\pi) = A^G$ and $\overline{A} = A/A^G$ is free, we have $A = A^G \oplus A'$ with $A' \cong \pi(A)$ via π. This decomposition induces a corresponding one for M, because $A^G \subseteq M$; so $M = A^G \oplus (M \cap A')$ and $M \cap A' \cong \pi(A) \cap \Lambda_+$ via π. This completes the proof of the theorem. $\square$

3.4. **Generators.** We now descibe how the foregoing leads to an explicit set of *fundamental invariants*, that is, algebra generators for R. Inasmuch as $R \cong k[M]$, this amounts to finding generators for M and tracing them through the isomorphism. As this isomorphism is the identity on $\mathrm{U}(M) = A^G$, we will concentrate on M_+.

3.4.1. *Generators for* $M_+ = \pi(A) \cap \Lambda_+$. Since the semigroup M_+ is positive, it has a unique minimal generating set, the so-called *Hilbert basis* of M_+. Here, in outline, is how to find this Hilbert basis; for complete details and an algorithmic treatment, see [21, Chapter 13].

Recall that $\Lambda_+ = \oplus_{i=1}^r \mathbb{Z}_+ \lambda_i$, where $\lambda_1, \ldots, \lambda_r$ are the fundamental dominant weights. These belong to $\pi(A) \otimes \mathbb{Q} \subseteq V$. Hence, there are suitable $0 \neq z_i \in \mathbb{Z}_+$ so that $m_i = z_i \lambda_i \in M_+$; we will assume that z_i is chosen minimal. The subset

$$K = \sum_{i=1}^r [0, m_i] = \{\sum_{i=1}^r t_i m_i \mid 0 \leq t_i \leq 1\}$$

of V is compact (a zonotope), and hence its intersection $K \cap M_+$ with the discrete M_+ is finite. It is easy to see that $K \cap M_+$ generates M_+; the Hilbert basis of M_+ can be found by selecting the indecomposable elements of $K \cap M_+$, that is, the elements $m \in K \cap M_+$ that cannot be written as $m = n + n'$ with $0 \neq n, n' \in K \cap M_+$. Note that $m_1, \ldots, m_r$ are certainly indecomposable, by the minimal choice of the z_i's and linear independence of the λ_i's. The remaining indecomposables in $K \cap M_+$ (if any) will be denoted $m_{r+1}, \ldots, m_s$; so $s \geq r = \mathrm{rank}(\overline{A})$.

We remark in passing that Gubeladze's polytope $\Phi(M_+)$ is the convex hull of $m_1, \ldots, m_r$ (up to projective equivalence; see (1.3)). Indeed, since $m_i \in M_+ = \mathbb{Z}_+(K \cap M_+) \subseteq \mathbb{R}_+\{m_1, \ldots, m_r\}$, we have $C(M_+) = \mathbb{R}_+\{m_1, \ldots, m_r\}$.

3.4.2. *Fundamental invariants.* As all $m_i \in \Lambda_+ = \oplus_{j=1}^r \mathbb{Z}_+ \lambda_j$, they have a unique representation of the form $m_i = \sum_j z_{i,j} \lambda_j$ with $z_{i,j} \in \mathbb{Z}_+$. For $i \leq r$, this representation is simply $m_i = z_i \lambda_i$, as above. Thus we obtain the following system of fundamental invariants:

$$\mu_i = \prod_{j=1}^r \sigma(\lambda_i)^{z_{i,j}} \ (i = 1, \ldots, s)$$

Here, $\mu_1 = \sigma(\lambda_1)^{z_1}, \ldots, \mu_r = \sigma(\lambda_r)^{z_r}$ are algebraically independent, as the $\sigma(\lambda_i)$'s are, and R is a finite module over the polynomial algebra $k[\mu_1, \ldots, \mu_r]$, since each μ_i, raised to a suitable power, belongs to $\langle \mu_1, \ldots, \mu_r \rangle$. In fact, since R is Cohen-Macaulay (cf. [3, Theorem 6.3.5]), R is a free module over $k[\mu_1, \ldots, \mu_r]$.

3.5. **The class group.** The formula given in [16] for the class group of R can be rewritten in terms of the above root system data. Indeed, by (2.3), $R = k[M] = k[\mathrm{U}(M)] \otimes \overline{R}$ is a Laurent polynomial extension of $\overline{R}$, and so $\mathrm{Cl}(R) = \mathrm{Cl}(\overline{R})$. Further, by [16], $\mathrm{Cl}(\overline{R}) = \mathrm{H}^1(G, \overline{A}^D)$, where D denotes the subgroup of G that is generated by those reflections that are *diagonalizable* on $\overline{A}$, that is, with respect to a suitable $\mathbb{Z}$-basis of $\overline{A}$, they have the form $\mathrm{diag}(-1, 1, \ldots, 1)$. Now G acts as a reflection group on $\overline{A}^D$, and the G-lattice $\overline{A}^D$ is effective, as $\overline{A}$ is. Thus, [15, Proposition 2.2.25] gives

$\mathrm{H}^1(G, \overline{A}^D) \cong \Lambda_{\overline{A}^D, G} / \overline{A}^D$. Hence,

$$\mathrm{Cl}(R) \cong \Lambda_{\overline{A}^D, G} / \overline{A}^D \ .$$

It is perhaps worth noting that $\Lambda_{\overline{A}^D, G} / \overline{A}^D$ is always a direct summand of $\Lambda_{A,G} / \pi(A) = \Lambda_{\overline{A}, G} / \overline{A}$. This follows from the fact that $\overline{A}^D$ is a direct summand of $\overline{A}$ as G-lattices; see [16, Lemma 2.4].

In the special case where A is effective at the outset and G contains no diagonalizable reflections, the above formula simplifies to

$$\mathrm{Cl}(R) \cong \Lambda / A \ ,$$

with $\Lambda = \Lambda_{A,G}$ as before.

Finally, we remark that the Picard group of R is trivial, as is in fact the full projective class group $K_0(R)/\langle[R]\rangle$. This is a consequence of Gubeladze's theorem [10] stating that all projective modules over $R = k[M]$ are free.

3.6. **Examples.** We illustrate the foregoing with a couple of explicit examples. In each case, A will be effective; so $\pi = \mathrm{Id}$ and $M = M_+ = A \cap \Lambda_+$. We will follow the notations in the proof of Theorem (3.3) and in (3.4) quite closely.

3.6.1. *An example in rank 2.* Let A be free abelian of rank 2, with $\mathbb{Z}$-basis $\{a, b\}$, and let G be the subgroup of $\mathrm{GL}(A) = \mathrm{GL}_2(\mathbb{Z})$ that is generated by the matrices $r = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $s = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. (These matrices act on the right on A, viewed as integer row vectors of length 2.) The generators r and s are reflections, and $G \cong S_3$, the symmetric group on 3 symbols. The only other reflection in G is $t = \left(\begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$; all reflections are conjugate in G, and none are diagonalizable. As a generator for $\mathrm{Ker}_A(g + \mathrm{Id})$, we choose $a_r = (-1, 1) = a^{-1}b$; similarly, we select $a_s = (0, 1) = b$ for s and $a_t = (1, 0) = a$ for t. So $\Phi = \{\pm a_r, \pm a_s, \pm a_t\}$ (a root system of type A_2). As base for Φ, we fix $\Delta = \{\alpha_1 = -a_t = (-1, 0), \alpha_2 = a_s = (0, 1)\}$; so $g_1 = t$ and $g_2 = s$. This leads to the fundamental dominant weights $\lambda_1 = (-2/3, 1/3)$, $\lambda_2 = (-1/3, 2/3)$. The zonotope $K = [0, m_1] + [0, m_2]$ of (3.4.1) is given by $m_i = 3\lambda_i$, and we obtain the following generators for M: m_1, m_2, and $m_3 = \lambda_1 + \lambda_2$.
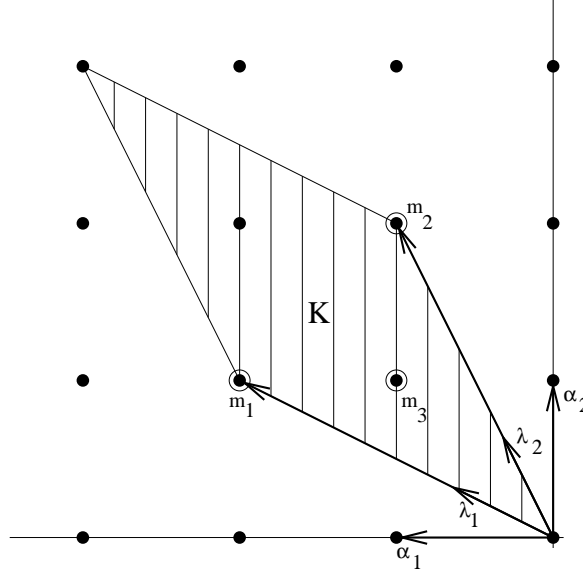
Therefore, $\sigma(\lambda_1)^3$, $\sigma(\lambda_2)^3$, and $\sigma(\lambda_1)\sigma(\lambda_2)$, form a fundamental system of invariants in R. Returning to multiplicative notation, the orbit sums for the fundamental dominant weights are:

$$\sigma(\lambda_1) = a^{-2/3}b^{1/3} + a^{1/3}b^{-2/3} + a^{1/3}b^{1/3} = a^{1/3}b^{1/3}(a^{-1} + b^{-1} + 1)$$

$$\sigma(\lambda_2) = a^{-1/3}b^{-1/3} + a^{-1/3}b^{2/3} + a^{2/3}b^{-1/3} = a^{-1/3}b^{-1/3}(a + b + 1) \ .$$

This leads to the following explicit system of fundamental invariants:

$$\mu_1 = \sigma(\lambda_1)^3 = ab(a^{-1} + b^{-1} + 1)^3,$$
$$\mu_2 = \sigma(\lambda_2)^3 = a^{-1}b^{-1}(a + b + 1)^3,$$
$$\mu_3 = \sigma(\lambda_1)\sigma(\lambda_2) = (a + b + 1)(a^{-1} + b^{-1} + 1) \ .$$

The class group of R evaluates to $\mathrm{Cl}(R) = \Lambda/A \cong \mathbb{Z}/3\mathbb{Z}$.

3.6.2. *Example in rank 3.* Let A be free abelian with $\mathbb{Z}$-basis $\{a, b, c\}$, and let G be the subgroup of $\mathrm{GL}(A) = \mathrm{GL}_3(\mathbb{Z})$ that is generated by the matrices $r = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $s = \left(\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{smallmatrix}\right)$, and $t = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. This group is isomorphic to S_4. The generators are reflections; they are all conjugate. The complete set reflections is the full G-conjugacy class: $\{r, s, t, w = r^t, u = s^t, v = s^w\}$; none are diagonalizable. The root system $\Phi = \Phi_{A,G}$ evaluates to $\Phi = \{\pm(1, 0, 0), \pm(1, 0, -1), \pm(1, -1, 0), \pm(0, 1, 0), \pm(0, 0, 1), \pm(0, 1, -1)\}$. A suitable base of Φ is $\Delta = \{\alpha_1 = a_t = (-1, 0, 1), \alpha_2 = a_r = (1, -1, 0), \alpha_3 = a_s = (0, 0, -1)\}$; so $g_1 = t$, $g_2 = r$, $g_3 = s$. This results in the following fundamental dominant weights: $\lambda_1 = (-1/2, -1/2, 1/2)$, $\lambda_2 = (1/4, -3/4, 1/4)$, and $\lambda_3 = (-1/4, -1/4, -1/4)$. The zonotope K is spanned by $m_1 = 2\lambda_1$, $m_2 = 4\lambda_2$, and

$m_3 = 4\lambda_3$, and the generators of M are: m_1, m_2, m_3, $m_4 = \lambda_2 + \lambda_3$, $m_5 = \lambda_1 + 2\lambda_2$, and $m_6 = \lambda_1 + 2\lambda_3$. Calculating the orbits sums:

$$\sigma(\lambda_1) = a^{-1/2}b^{-1/2}c^{-1/2}(a + b + c + ab + ac + bc),$$

$$\sigma(\lambda_2) = a^{1/4}b^{1/4}c^{1/4}(a^{-1} + b^{-1} + c^{-1} + 1),$$

$$\sigma(\lambda_3) = a^{-1/4}b^{-1/4}c^{-1/4}(a + b + c + 1).$$

This leads to the following explicit system of fundamental invariants:

$$\mu_1 = \sigma(\lambda_1)^2 = a^{-1}b^{-1}c^{-1}(a + b + c + ab + ac + bc)^2,$$

$$\mu_2 = \sigma(\lambda_2)^4 = abc(a^{-1} + b^{-1} + c^{-1} + 1)^4,$$

$$\mu_3 = \sigma(\lambda_3)^4 = a^{-1}b^{-1}c^{-1}(a + b + c + 1)^4,$$

$$\mu_4 = \sigma(\lambda_2)\sigma(\lambda_3) = (a + b + c + 1)(a^{-1} + b^{-1} + c^{-1} + 1),$$

$$\mu_5 = \sigma(\lambda_1)\sigma(\lambda_2)^2 = (a + b + c + ab + ac + bc)(a^{-1} + b^{-1} + c^{-1} + 1)^2,$$

$$\mu_6 = \sigma(\lambda_1)\sigma(\lambda_3)^2 = (a^{-1} + b^{-1} + c^{-1} + a^{-1}b^{-1} + a^{-1}c^{-1} + b^{-1}c^{-1})(a + b + c + 1)^2.$$

For the class group of R, we obtain $\mathrm{Cl}(R) = \Lambda/A \cong \mathbb{Z}/4\mathbb{Z}$.

The calculations for this example were performed with GAP (version 3.4) [19]; the code is available under http://www.math.temple.edu/~lorenz/semigroup.html.

4. FIXED-POINT-FREE ACTIONS

We continue with the notation of (2.1). In addition, we assume in this section that char k *does not divide the order of* G. *Finally, we will continue to assume that* G *acts faithfully on* A; *so* $G \subseteq \mathrm{GL}(A)$.

4.1. **Cotangent spaces.** The (Zariski) *cotangent space* at the maximal ideal $\mathfrak{M}$ of S is the k-space

$$\mathfrak{M}/\mathfrak{M}^2 .$$

It is a $k[G^T(\mathfrak{M})]$-module, where $G^T(\mathfrak{M}) = \{g \in G \mid s - s^g \in \mathfrak{M} \text{ for all } s \in S\}$ is the inertia group of $\mathfrak{M}$.

Lemma. *Assume k algebraically closed. Then, for each maximal ideal $\mathfrak{M}$ of S, there is an isomorphism of $k[G^T(\mathfrak{M})]$-modules $A \otimes_\mathbb{Z} k \xrightarrow{\cong} \mathfrak{M}/\mathfrak{M}^2$. An element $g \in G^T(\mathfrak{M})$ is a pseudoreflection on $\mathfrak{M}/\mathfrak{M}^2$ if and only if g is a reflection on A.*

Proof. Let μ be the k-algebra homomorphism $\mu : S \twoheadrightarrow S/\mathfrak{M} \xrightarrow{\cong} k$. The $k[G^T(\mathfrak{M})]$-isomorphism is given by

$$\begin{aligned} A \otimes_\mathbb{Z} k &\xrightarrow{\cong} & \mathfrak{M}/\mathfrak{M}^2 \\ a \otimes 1 &\longmapsto & \mu(a)^{-1}a - 1 + \mathfrak{M}^2 & (a \in A) . \end{aligned}$$

Clearly, an element $g \in G^T(\mathfrak{M})$ is a pseudoreflection on $\mathfrak{M}/\mathfrak{M}^2$ if and only if g acts as a pseudoreflection on $A \otimes_\mathbb{Z} k$. Thus, it suffices to show that $g \in G$ acts as a

(pseudo)reflection on A if and only if g does so on $V = A \otimes_{\mathbb{Z}} k$. This is a consequence of the following more general equality for g-fixed point sets:

$$(3) \qquad\qquad V^{\langle g \rangle} = A^{\langle g \rangle} \otimes_{\mathbb{Z}} k \text{ holds for all } g \in G.$$

The inclusion $\supseteq$ being clear, we proceed to prove $\subseteq$. First, $V^{\langle g \rangle} = V_0^{\langle g \rangle} \otimes_{k_0} k$, where $V_0 = A \otimes_{\mathbb{Z}} k_0$ and k_0 denotes the prime subfield of k. If $k_0 = \mathbb{Q}$, then clearly $V_0^{\langle g \rangle} = A^{\langle g \rangle} \otimes_{\mathbb{Z}} \mathbb{Q}$. So assume that $k_0 = \mathbb{F}_p$. Then the $\langle g \rangle$-cohomology sequence that is associated with $A \xrightarrow{p} A \twoheadrightarrow V_0 = A/pA$ in conjunction with the fact that $\mathrm{H}^1(\langle g \rangle, A/pA)$ is trivial (because $p = \operatorname{char} k$ does not divide the order of g) proves that $A^{\langle g \rangle}$ maps onto $V_0^{\langle g \rangle}$, which finishes the proof. $\qquad\square$

4.2. **Singularities.** The *singular locus* of R is defined by

$$\operatorname{Sing}(R) = \{ \mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{gldim}(R_{\mathfrak{p}}) = \infty \} \ ;$$

it is a closed subset of $\operatorname{Spec}(R)$ of codimension at least 2 (e.g., [14, Chapt. VI]). The complement will be denoted $\operatorname{Reg}(R)$.

Lemma. *Assume k is algebraically closed. Let $\mathfrak{M}$ be a maximal ideal of S and put $\mathfrak{m} = \mathfrak{M} \cap R$, a maximal ideal of R. Then $\mathfrak{m} \in \operatorname{Reg}(R)$ if and only if $G^T(\mathfrak{M})$ is a reflection group on A.*

Proof. In view of Lemma (4.1), this is immediate from the following criterion of Serre [20] (cf. also [1, Exercise 7 on p. 138]):

$\mathfrak{m} \in \operatorname{Reg}(R)$ iff $G^T(\mathfrak{M})$ acts as a pseudoreflection group on $\mathfrak{M}/\mathfrak{M}^2$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In case G acts without reflections on A, the foregoing leads to a particularly manageable description of $\operatorname{Sing}(R)$. For this, we put

$$I = \bigcap_{1 \neq g \in G} I(g) \quad \text{with} \quad I(g) = (s - s^g \mid s \in S)S \ .$$

The ideal I is G-stable and semiprime, with

$$(4) \qquad\qquad \operatorname{height} I = \min_{1 \neq g \in G} \operatorname{rank}(1 - g)_A$$

(see [18, Lemma 3.2] and [2, 2.6]). So G acts without reflections on A if and only if $\operatorname{height} I \geq 2$.

Corollary. *Assume that $G \neq \langle 1 \rangle$ acts without reflections on A. Then, via Lying Over,*

$$\operatorname{Sing}(R) \xleftrightarrow{\ 1-1\ } \{ \mathfrak{M} \in \operatorname{Spec}(S) \mid \mathfrak{M} \supseteq I \}/G \ .$$

This set contains at least two elements.

Proof. Recall that Lying Over yields a one-to-one correspondence of $\mathrm{Spec}(R)$ with $\mathrm{Spec}(S)/G$, the set of G-orbits in $\mathrm{Spec}(S)$: The primes $\mathfrak{p}$ of R are exactly the ideals of the form $\mathfrak{p} = \mathfrak{P} \cap R$, where $\mathfrak{P}$ is a prime of S, said to "lie over" $\mathfrak{p}$, and the full set of primes of S lying over a particular prime of R forms a G-orbit.

Now let $\mathfrak{m}$ be a maximal ideal of R and let $\mathfrak{M}$ be a maximal ideal of S lying over $\mathfrak{m}$. Then, by the above Lemma, $\mathfrak{m} \in \mathrm{Reg}(G)$ if and only if $G^T(\mathfrak{M}) = \langle 1 \rangle$. In other words, since $g \in G^T(\mathfrak{M})$ is equivalent with $\mathfrak{M} \subseteq I(g)$, we have

$$\mathfrak{m} \in \mathrm{Sing}(R) \iff \mathfrak{M} \subseteq I .$$

An arbitrary prime $\mathfrak{p}$ of R is the intersection of all maximal ideals $\mathfrak{m} \supseteq \mathfrak{p}$, and $\mathfrak{p}$ belongs to $\mathrm{Sing}(R)$ precisely if all these $\mathfrak{m}$'s do. This implies the description of $\mathrm{Sing}(R)$.

The kernel of the distinguished augmentation of $S = k[A]$ is a G-stable maximal ideal of S containing I, and hence it accounts for a point in $\mathrm{Sing}(R)$. If it was the only point, then $S/I = k$. But, for any element $g \in G$ of prime order, $\det(1 - g)$ is divisible by the same prime, and so $A^{1-g} \neq A$. Therefore, $S/I(g) \cong k[A/A^{1-g}] \neq k$, and so $S/I \neq k$. $\square$

4.3. **A negative result.** The group G is said to act *fixed point freely* on A if $A^{\langle g \rangle}$ is trivial for all $1 \neq g \in G$. By (4.2)(4), this is equivalent with $\mathrm{height}\, I = \mathrm{rank}(A)$, which in turn just says that S/I is finite k-dimensional. Therefore, as long as G acts without reflections on A and k is algebraically closed, Corollary (4.2) implies that

$\mathrm{Sing}(R)$ *is finite if and only if G acts fixed point freely on A.*

This observation will be used in the proof of the following

Theorem. *If G acts fixed point freely on $\overline{A} = A/A^G$ and $\mathrm{rank}(\overline{A}) \geq 2$ then R is not a semigroup algebra.*

Proof. Suppose, by way of contradiction, that R is a semigroup algebra. Then, by Proposition (2.3), so is $\overline{R}$. Thus we mave assume that G does in fact act fixed point freely on A. Furthermore, extending scalars if necessary, we may assume k to be algebraically closed.

As we have remarked above, $\mathrm{Sing}(R)$ is finite. On the other hand, R is a semigroup algebra, say $R \cong k[M]$. Necessarily, M is an affine normal semigroup with trivial group of units, as A^G is trivial. Thus, the action of the torus T on $\mathrm{Max}\, R$, as described in (1.4), has exactly one fixed point. This action stabilizes $\mathrm{Sing}(R)$. Since T is connected and $\mathrm{Sing}(R)$ is finite, T must act trivially on $\mathrm{Sing}(R)$. We conclude that in fact $\#\,\mathrm{Sing}(R) = 1$, contradicting Corollary (4.2). This finishes the proof. $\square$

Since effective lattices for groups of prime order are clearly fixed point free, we obtain the following

Corollary. *If G has odd prime order then R is not a semigroup algebra.*

4.4. **An example.** If rank $A = 2$ then finite subgroups of $\mathrm{GL}(A)$ either act fixed point freely or else they are generated by reflections; see, e.g., [16, 2.7]. This is of course no longer true in higher ranks. Here we discuss a specific example which is not directly covered by the foregoing. Nevertheless, a look at the singularities very much like the proof of Theorem (4.3) still yields the desired conclusion. The example is taken from [13], where it was used for different (very interesting) purposes.

For a given n, let $A = \langle a_1 \rangle \times \ldots \times \langle a_n \rangle$ be free abelian of rank n, and let $G = \mathrm{diag}(\pm 1, \ldots, \pm 1)_{n \times n} \cap \mathrm{SL}(A)$. So G contains no reflections but does not act fixed point freely if $n > 2$. We assume k algebraically closed with char $k \neq 2$.

It is not hard to check that the algebra R of multiplicative G-invariants has the presentation

$$R \cong k[x_1, \ldots, x_n, z] / \left(z^2 - \prod_{i=1}^{n} (x_i^2 - 1) \right) .$$

Using the Jacobian criterion of [14, p. 173], one checks that $\mathrm{Sing}\, R$ is the union of $4\binom{n}{2}$ affine spaces $\mathbb{A}_k^{n-2}$, the irreducible components of $\mathrm{Sing}\, R$.

Alternatively, whenever G has no reflections, such as in our example, the irreducible components of $\mathrm{Sing}\, R$ correspond to the G-orbits of the minimal primes over the ideal I in Corollary (4.2). Here, one calculates easily that

$$I = \bigcap_{i \neq j \in \{1, \ldots, n\}} P_{i,j}^{\pm, \pm} \qquad \text{with} \qquad P_{i,j}^{\pm, \pm} = (a_i \pm 1, a_j \pm 1)S .$$

The $P_{i,j}^{\pm, pm}$ are the minimal primes over I; they are all G-invariant, and hence they correspond to the irreducible components of $\mathrm{Sing}\, R$. Moreover, $R/R \cap P_{i,j}^{\pm,\pm} = \left(S/P_{i,j}^{\pm, pm} \right)^G = k[A_{i,j}]^G$, where $A_{i,j}$ denotes the sublattice of A that is spanned by all a's except for a_i and a_j. Since G acts on $A_{i,j}$ as the full group $\mathrm{diag}(\pm 1, \ldots, \pm 1)_{n-2 \times n-2}$, it is easy to see that $k[A_{i,j}]^G$ is a polynomial algebra of dimension $n - 2$. Thus the irreducible components of $\mathrm{Sing}\, R$ are affine $(n-2)$-spaces.

Now assume R is a semigroup algebra. Then, by the connectedness argument used in the proof of Theorem (4.3), the torus action considered there stabilizes all irreducible components of $\mathrm{Sing}\, R$, and hence also all their intersections. In the present case, the minimal nonempty intersections are a collection of 2^n points corresponding to the maximal ideals $(a_1 \pm 1, \ldots, a_n \pm 1)S \cap R$ of R. Thus all these are torus fixed points, while there can only be one. This contradiction shows that, again, R is not a semigroup algebra.

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